O.D.E. (Unit - III)

System of Homogeneous ODE's of Ist order and Ist degree with Constant Co-efficients

$$\dot{x}_1 = \frac{dx_1}{dt} = 3x_1 + 4x_2$$

$$\dot{x}_2 = \frac{dx_2}{dt} = 2x_1 + 3x_2$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{--- (1)}$$

$$\dot{x} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} x \quad \text{--- (2)} \qquad \text{where, } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x$$

$$\dot{x} = ax$$

$$\Rightarrow \frac{dx}{dt} = ax$$

$$\Rightarrow \frac{dx}{x} = a\,dt$$

$$\log x = at + \underbrace{\log v}_{\text{constant}}$$

$$\Rightarrow \log\left(\frac{x}{v}\right) = at$$

$$\Rightarrow x = v e^{at}$$

where, v and a are Constants

We seek the solution of (2) in the form

$x_1 = v_1 e^{\lambda t}$

$x_2 = v_2 e^{\lambda t}$

ie. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$ —— (3)

putting in eq^n (1), we get

$$\frac{d}{dt}\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} \right) = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$$

$$\Rightarrow \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cancel{e^{\lambda t}} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cancel{e^{\lambda t}}$$

$$\Rightarrow \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \text{—— (4)}$$

For non-zero eigen vector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\begin{vmatrix} 3-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

on solving, we get

$\lambda = 3 \pm 2\sqrt{2}$

$\therefore$ Sol^n of the given system of diff. eq^n is

$$x = c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda_2 t}$$

To find the solution of $\dot{x} = Ax$
where, $x = [x_1, x_2, x_3, \cdots x_n]^T$
A is an $n \times n$ matrix.

Case I:
A has distinct eigen values.

Step-I:
Find all the eigenvalues of A by solving the polynomial equation
$$|A - \lambda I| = 0$$

Step II:
Find all the eigenvectors by solving
$$(A - \lambda I)v = 0$$

Corresponding to n eigenvalues $\lambda_1, \lambda_2 \cdots \lambda_n$ there will be n linearly independent eigenvectors $[v_{11}, v_{12}, \cdots, v_{1n}]^T$, $[v_{21}, v_{22}, \cdots v_{2n}]^T$ ...... $[v_{n1}, v_{n2}, \cdots, v_{nn}]^T$.

Step – III

The complete solution of the given system of differential equations is

$$x = c_1 \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{bmatrix} e^{\lambda_2 t} + \cdots$$

$$\cdots + c_n \begin{bmatrix} v_{n1} \\ v_{n2} \\ \vdots \\ v_{nn} \end{bmatrix} e^{\lambda_n t}$$

Remark:

If $\begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{bmatrix}$ is an eigenvector

then $\begin{bmatrix} a\,v_{i1} \\ a\,v_{i2} \\ \vdots \\ a\,v_{in} \end{bmatrix}$ is also an eigenvector.

Q). Solve

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} x$$

<u>Solution:</u>

<u>Step I</u>: Eigenvalues of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & -11 & 6-\lambda \end{vmatrix} = 0$$

$$C_1 \longrightarrow C_1 + C_2 + C_3$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1-\lambda & -\lambda & 1 \\ 1-\lambda & -11 & 6-\lambda \end{vmatrix} = 0$$

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$$\Rightarrow (1-\lambda)\begin{vmatrix} 1 & 1 & 0 \\ 1 & -\lambda & 1 \\ 1 & -11 & 6-\lambda \end{vmatrix} = 0$$

$R_2 \rightarrow R_2 - R_1$
$R_3 \rightarrow R_3 - R_1$

$$(1-\lambda)\begin{vmatrix} 1 & 1 & 0 \\ 0 & -\lambda-1 & 1 \\ 0 & -12 & 6-\lambda \end{vmatrix} = 0$$

or, $(1-\lambda)\ 1\{(-\lambda-1)(6-\lambda)+12\} = 0$

or, $(1-\lambda)\{-6\lambda+\lambda^2-6+\lambda+12\} = 0$

or, $(1-\lambda)(\lambda^2-5\lambda+6) = 0$

or $(1-\lambda)(\lambda-2)(\lambda-3) = 0$

$\lambda = 1, 2, 3$ —— (1)

all are distinct eigenvalues.

Step - II

$$(A - \lambda I) v = 0$$

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & -11 & 6-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (2)}$$

Eigenvector corresponding to eigenvalue $\lambda = 1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-v_1 + v_2 = 0$$
$$-v_2 + v_3 = 0$$

$\Rightarrow$ $v_1 = v_2$
and $v_2 = v_3$.

i.e, $v_1 = v_2 = v_3 = K$ (say) $= 1$ (say)

Eigenvector corresponding to eigenvalue
$\lambda = 2$

$\therefore$ from (2)

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2v_1 + v_2 = 0$$
$$-2v_2 + v_3 = 0$$

$\Rightarrow \quad 2v_1 = v_2$

and $\quad 2v_2 = v_3$

Let us take $v_1 = 1$.
then $v_2 = 2$ and $v_3 = 4$

Eigenvector corresponding to $\lambda = 3$

$$\begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 6 & -11 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -3v_1 + v_2 = 0$

$\quad -3v_2 + v_3 = 0$

$\Rightarrow \quad 3v_1 = v_2$

and $3v_2 = v_3$.

Taking $v_1 = 1$

we get, $v_2 = 3$ and $v_3 = 9$

Step- III.

The complete solution is.

$$x = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{1t} + c_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} e^{3t}$$